Inequality 7

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AM-GM Inequality :-

$a, b \in \mathbb{R}^+$, $\frac{a+b}{2} \geq \sqrt{ab}$

$a_1, a_2, \ldots, a_n \in \mathbb{R}^+$, then,

$$\boxed{\frac{a_1+a_2+a_3+\cdots+a_n}{n} \geq \sqrt[n]{a_1 a_2 \cdots a_n}}$$

Proof:- By Induction, for $n=1$ it is true $\rightarrow P_1$

for $n=2$ we get $AM \geq GM$ true $\rightarrow P_2$

Let P_n denote that $AM \geq GM$ for n numbers.

We need to prove (a) $P_n \Rightarrow P_{n-1}$ and (b) $P_n \Rightarrow P_{2n}$

(a) $a_1, a_2, \ldots, a_{n-1} \in \mathbb{R}^+$ and let $g = \sqrt[n-1]{a_1 \cdots a_{n-1}}$ $\Rightarrow g^{n-1} = a_1 a_2 \cdots a_n$

we have P_n as $\frac{a_1+a_2+\cdots+a_{n-1}+g}{n} \geq \sqrt[n]{a_1 a_2 \cdots a_{n-1} g} = \sqrt[n]{g^{n-1} g} = g$

$$a_1+a_2+\cdots+a_{n-1}+g \geq ng$$

$$\Rightarrow a_1+a_2+\cdots+a_{n-1} \geq (n-1)g$$

$$\Rightarrow \frac{a_1+a_2+\cdots+a_{n-1}}{n-1} \geq g = \sqrt[n-1]{a_1 a_2 \cdots a_{n-1}} \rightarrow P_{n-1}$$

So we proved $P_n \Rightarrow P_{n-1}$

$a_1+a_2 \geq 2\sqrt{a_1a_2}$

$a_3+a_4 \geq 2\sqrt{a_3a_4}$

$\Downarrow$

(b) $a_1, a_2, \ldots, a_{2n} \in \mathbb{R}^+$, then,

$$a_1+a_2+\cdots+a_{2n} = (a_1+a_2)+(a_3+a_4)+\cdots+(a_{2n-1}+a_{2n})$$

we have P_2 true $\Rightarrow$ $\geq 2\left(\sqrt{a_1a_2}+\sqrt{a_3a_4}+\cdots+\sqrt{a_{2n-1}a_{2n}}\right)$

we have P_n true $\Rightarrow$ $\geq 2n\left(\sqrt{a_1a_2}\sqrt{a_3a_4}\cdots\sqrt{a_{2n-1}a_{2n}}\right)^{1/n}$

$$\Rightarrow \frac{a_1+a_2+\cdots+a_{2n}}{2n} \geq \sqrt[2n]{a_1a_2\cdots a_{2n-1}a_{2n}}$$

So we get $P_n \Rightarrow P_{2n}$

So we get General AM $\geq$ GM as true

(*) Another proof will be discussed when proof writing will be discussed.

Q> Find the maximum value of $x(1-x^3)$ for $0 \leq x \leq 1$

Ans:-

$$\left.\begin{aligned} 0 &\geq -x \geq -1 \\ 0 &\geq -x^3 \geq -1 \\ 1 &\geq 1-x^3 \geq 0 \\ 1 &\geq x \geq 0 \\ 1 &\geq x(1-x^3) \geq 0 \end{aligned}\right\}$$

This way we will only get a bound, not a maximum value

(*) <u>Idea</u>:- The idea of the proof is to exchange the product for another one in such a way that the sum of the elements becomes constant.

$$y = x(1-x^3)$$

$$3y^3 = \underline{3x^3}\,\underline{(1-x^3)}\,\underline{(1-x^3)}\,\underline{(1-x^3)} \longrightarrow (GM)^4$$

Then $\text{AM} = \dfrac{3x^3 + (1-x^3) + (1-x^3) + (1-x^3)}{4}$

$$= \frac{3}{4}$$

$$\text{AM} \geq \text{GM} \Rightarrow (\text{AM})^4 \geq (\text{GM})^4 \Rightarrow \left(\frac{3}{4}\right)^4 \geq 3y^3$$

$$\Rightarrow y^3 \leq \left(\frac{3}{4}\right)^4 \frac{1}{3}$$

$$\Rightarrow y \leq \frac{3}{4\sqrt[3]{4}}$$

$$\Rightarrow x(1-x^3) \leq \frac{3}{4\sqrt[3]{4}}$$

Maximum value is reached when all elements of the AM $\geq$ GM are equal. The value of x at which maximum value is reached is

$$3x^3 = (1-x^3) \Rightarrow 4x^3 = 1 \Rightarrow x^3 = \frac{1}{4} \Rightarrow x = \frac{1}{\sqrt[3]{4}}$$

are equal. The value of x at which

$$3x^3 = (1-x^3) \Rightarrow 4x^3 = 1 \Rightarrow x^3 = \frac{1}{4} \Rightarrow x = \frac{1}{\sqrt[3]{4}}$$

Q> For $x, y \in \mathbb{R}$ prove that $x^4 + y^4 + 8 \geq 8xy$

Ans:- $\dfrac{x^4+y^4+4+4}{4} \geq \sqrt[4]{4 \cdot 4 \cdot x^4 y^4}$ $\Leftarrow$ AM-GM inequality

$$\Rightarrow x^4+y^4+8 \geq 4 \cdot 2^{4/4} xy = 8xy$$

Q> For $a, b, c, d \in \mathbb{R}^+$ prove that $(a+b+c+d)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}\right) \geq 16$

Ans:- $a+b+c+d \geq 4\sqrt[4]{abcd}$

$\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d} \geq 4\sqrt[4]{\frac{1}{abcd}}$

$\Rightarrow (a+b+c+d)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}\right) \geq 16$

Q> For $a, b, c, d \in \mathbb{R}^+$, prove that $\frac{a}{b}+\frac{b}{c}+\frac{c}{d}+\frac{d}{a} \geq 4$

Ans:- $\frac{a}{b}+\frac{b}{c}+\frac{c}{d}+\frac{d}{a} \geq 4\sqrt[4]{\frac{a}{b}\frac{b}{c}\frac{c}{d}\frac{d}{a}} = 4$

Q> For $x, y, z \in \mathbb{R}$, $x^2+y^2+z^2 \geq x\sqrt{y^2+z^2} + y\sqrt{x^2+z^2}$

Ans:- $\dfrac{x^2+(y^2+z^2)}{2} \geq x\sqrt{y^2+z^2}$

$\dfrac{(x^2+z^2)+y^2}{2} \geq y\sqrt{x^2+z^2}$

$\Rightarrow x^2+y^2+z^2 \geq x\sqrt{y^2+z^2} + y\sqrt{x^2+z^2}$

Q> For $x, y, z \in \mathbb{R}^+$, prove that $\frac{xy}{z}+\frac{yz}{x}+\frac{zx}{y} \geq x+y+z$

Ans:- $\dfrac{\frac{xy}{z}+\frac{yz}{x}}{2} \geq y$, $\dfrac{\frac{yz}{x}+\frac{zx}{y}}{2} \geq z$, $\dfrac{\frac{xy}{z}+\frac{zx}{y}}{2} \geq x$

$$\Rightarrow \frac{xy}{z} + \frac{yz}{x} + \frac{zx}{y} \geq x+y+z$$

Q> Let $x_i > 0$, $i = 1, \ldots, n$. Prove that,

$$(x_1 + x_2 + \cdots + x_n)\left(\frac{1}{x_1} + \frac{1}{x_2} + \cdots + \frac{1}{x_n}\right) \geq n^2$$

Ans:- $x_1 + x_2 + \cdots + x_n \geq n\sqrt[n]{x_1 x_2 \cdots x_n}$ $\Rightarrow (x_1 + x_2 + \cdots + x_n)\left(\frac{1}{x_1} + \frac{1}{x_2} + \cdots + \frac{1}{x_n}\right) \geq n^2$

$\frac{1}{x_1} + \frac{1}{x_2} + \cdots + \frac{1}{x_n} \geq n\sqrt[n]{\frac{1}{x_1 x_2 \cdots x_n}}$

Q> If $a > 1$, then, prove $a^n - 1 > n\left(a^{\frac{n+1}{2}} - a^{\frac{n-1}{2}}\right)$

Ans:- HomeWork

Q> If $a, b, c > 0$ and $(1+a)(1+b)(1+c) = 8$ then prove that $abc \leq 1$

Ans:- HomeWork

Q> $a_0 = 1$, $a_1 = 1$, $a_n = a_{n-1} a_{n-2} + 1$, $n > 1$

a_{465} and a_{466}

$a_2 =$ odd × odd + 1 = even

$a_3 =$ even × ( ) + 1 = odd

$a_4 =$ odd

$a_5 =$ even

$a_6 =$ odd

$a_7 =$ odd